Approximation of Multiple Integrals over Hyperboloids with Application to a Quadratic Portfolio with Options.

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Abstract

We consider an application involving a financial quadratic portfolio of options, when the joint underlying log-returns changes with multivariate elliptic distribution. This motivates the need for methods for the approximation of multiple integrals over hyperboloids. A transformation is used to reduce the hyperboloid integrals to a product of two radial integrals and two spherical surface integrals. Numerical approximation methods for the transformed integrals are constructed. The application of these methods is demonstrated using some financial applications examples.

1 Introduction

Value-at-Risk (VaR) is considered to be one of the standard measures of market risk. VaR measures the maximum loss that a portfolio can experience with a certain probability over a certain horizon, for example, one day. Mathematically, if the profit or loss is given by $\Pi(t) - \Pi(0)$, for which $\Pi(t)$ is the price of the portfolio at t, VaR for a confidence level 1-α is determined by the following equation:

$$\mathbb{P}rob\{\Pi(0, S(0)) - \Pi(t, S(t)) > VaR_\alpha\} = \alpha,$$

where $S(t) = (S_1(t), \ldots, S_n(t))$ is a vector of asset prices that govern risk factors.

In this paper, we consider numerical methods for the estimation of integrals over hyperboloids and their application to VaR computations for a complex portfolio that contains options depending on the market fluctuations that create risk. We reduce the problem of VaR computations to multiple integrals over hyperboloids, and show how these integrals can be approximated using techniques described by Genz and Monahan [7] and Sheil and O'Muircheartaigh[11].

One of the most important analytic methods for VaR computation, which is called Δ-normal VaR, was introduced in the RiskMetrics Technical Document (1996). The method is based on the assumptions that the distribution is Normal and the portfolio is linear. Sadefo-Kamdem [13], generalized the Δ-Normal VaR by introducing the Δ-elliptic VaR for a linear portfolio, with the Δ-Student VaR given as an example. An advantage of the Δ-elliptic VaR (for example, Δ-Normal or Δ-Student) is that the formula is still fairly

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simple to calculate. But in practice, if we deal with a Δ-hedged portfolio, the Δ-elliptic VaR does not provide a realistic model, and that is why alternatives were proposed in the paper of Brummelhuis, Cordoba, Seco and Quintanilla [2]. In that paper, mathematical stationary phase techniques were used to estimate VaR for a quadratic portfolio when the risk factor changes with a Normal distribution. In a sequel paper, Brummelhuis and Sadefo-Kamdem [3] provided an estimation method with more precision for a VaR with a quadratic portfolio and generalized Laplace distributions.

We assume, as in [3], that the approximation for the price of the portfolio is given by

$$\Pi(t) - \Pi(0) \approx \Theta t + \Delta \mathbb{X}^t + \frac{1}{2}\mathbb{X}^t \Gamma \mathbb{X},$$

and we also assume that the joint log-returns $\mathbb{X}$ is elliptically distributed. For further details about elliptic distributions, see Embrechts, McNeil and Straussman [5]. Γ, Θ and Δ are functions of some sensitivities of the portfolio (see Taleb [14], 1997, for a discussion concerning sensitivities). We also suppose that $t = 1$, because the time horizon for VaR is generally taken to be one day If the log-returns of $\mathbb{X}$ are Normally distributed, Albanese and Seco [1] have shown how to reduce the analysis of a quadratic VaR to the computation of the integral of a Gaussian over a quadric in a space of possibly very high dimension. We will use the more general assumption of an elliptic distribution for the risk factors.

This paper proposes numerical methods for the approximation of integrals over hyperboloids, with application to estimate the VaR. We combine some techniques described in [6] ,[7] and [11] to approximate the integrals over hyperboloids for VaR, with the generalized assumption that the underlying joint log-returns changes with an elliptic distribution. To illustrate our method, we will take the familiar case of Normal distributions, and we consider test examples for two Δ-hedged portfolios from the French CAC 40 market. Brummelhuis, Cordoba, Quintanilla and Seco [2] have considered a quadratic portfolio with an analytic approximation for the Gaussian integrals over quadrics. Sadefo-Kamdem and Brummelhuis [3] have provided a similar analysis with a Generalized Laplace distribution. Albanese and Seco [1] investigated the approximation of integral of a Gaussian over a hyperboloid region with Fourier transform methods.

One of the most common methods for quadratic perturbations of the linear VaR uses the Cornish-Fisher expansion for the quantile function of non-Gaussian variables. There are also some quadratic approximations in Hull [8] and Dowd [4]. Many papers in literature have proposed numerical methods for the quadratic approximation (see, for example, [12], where Sadefo-Kamdem proposed the use of some numerical methods of Genz, [6], and the use of hypergeometric functions for a portfolios of equities VaR with multivariate t-Student distribution).

The rest of our paper is organized as follows. In Section 2 and 3, following Albanese and Seco [1], we show how portfolio volatility can be used to reduce the calculation of VaR to the approximation of integral over hyperboloid, assuming elliptic distributions that admit a density function. In section 4, we propose a numerical method for the approximation of integrals over hyperboloids using some methods of Genz [6], Genz and Monahan [7] and [11]. To illustrate our method we use examples where the density function is Normal. In section 5, we consider two examples of financial portfolios, and we have showed that our method is applicable to estimate the VaR for the portfolio. In Section 6 , we provide some conclusions.

2 Quadratic Portfolio of Options Application

In this section, we will define a quadratic portfolio of options as Quintanilla did in [10]. We first define $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_n)$, with

$$\mathbb{X}_i = log(S_i(t)/S_i(0)),$$

and we define $\Delta_1 = (\Delta_1^1, \ldots, \Delta_1^n)$, with $\Delta_1^i = S_i(0).\Delta^i$, and $\Delta = (\Delta^1, \ldots, \Delta^n)$, the gradient vector of the portfolio at time $t = 0$. We also define $\Gamma_1 = \left(\Gamma_1^{i,j}\right)_{i,j=1,\ldots,n}$ by

$$\Gamma_1^{i,j} = \begin{cases} S_i^2(0)\Gamma^{i,i} + \Delta_1^i & \text{if } i = j \\ S_i(0)S_j(0) \cdot \Gamma^{i,j} & \text{if } i \neq j \end{cases},$$

with $\Gamma = \left(\Gamma^{i,j}\right)_{i,j=1,\ldots,n} = \left(\frac{\partial^2 \Pi}{\partial S_i \partial S_j}(0)\right)_{i,j=1,\ldots,n}$, the Hessian of the portfolio at time $t = 0$. If we use a 2^{nd} order Taylor series approximation for Π, then

$$\Pi(t, S(t)) - \Pi(0, S(0)) \approx t\Theta + \Delta_1 \mathbb{X}^t + \mathbb{X}\Gamma_1 \mathbb{X}^t/2,$$

where $\Theta = \frac{\partial \Pi}{\partial t}(0)$.

If we consider a Δ-hedged Portfolio, we have $\Delta = 0$, and therefore $\Delta_1 = 0$. Our goal is to determinate the Value-at-Risk quantity V with confidence level $1 - \alpha$ when $t = 1$, as a solution to the equation

$$G_{\Gamma_1}(-V) = \mathbb{P}(\Theta + \mathbb{X}\Gamma_1 \mathbb{X}^t/2 \leq -V) = \alpha.$$

If we assume that the joint underlying log-returns $\mathbb{X}$ have a multivariate elliptic distribution with zero mean, then $G_{\Gamma_1}(-V)$ is given by

$$G_{\Gamma_1}(-V) = \int_{\{\Theta + x\Gamma_1 x^t/2 \leq -V\}} g(x\Sigma^{-1}x^t)\frac{dx}{\sqrt{det(\Sigma)}} = \alpha.$$

3 Transformation to a Hyperboloid Integration Region

We first decompose Σ as $\Sigma = \mathbb{C}\mathbb{C}^t$, where $\mathbb{C}$ is the (lower-triangular) Cholesky decomposition factor of Σ, and then we use the transformation $x = y\mathbb{C}^t$ to give

$$G_{\Gamma}(-V) = \int_{\{\Theta + y\mathbb{C}^t\Gamma\mathbb{C}y^t/2 \leq -V\}} g(yy^t)dy.$$

We next assume that the sensitivity-adjusted variance-covariance matrix, $\mathbb{C}^t\Gamma\mathbb{C}$, has a diagonalization in the form $\mathbb{C}^t\Gamma\mathbb{C} = \mathbb{O}\mathbb{D}\mathbb{O}^t$, with $\mathbb{O}$ orthogonal, and $\mathbb{D}$ diagonal. $\mathbb{C}\Gamma\mathbb{C}^t$ is not necessarily definite, but we can assume that a diagonalization has been constructed with

$$\mathbb{D} = \begin{pmatrix} D_+ & 0 \\ 0 & -D_- \end{pmatrix},$$

where

$$D_\epsilon = \begin{pmatrix} d_1^\epsilon & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & d_{n_\epsilon}^\epsilon \end{pmatrix}$$

for $\epsilon = \pm 1$, and where all $d_j^+, d_j^- \geq 0$, and $-d_1^- \leq -d_2^- \leq \ldots \leq -d_{n_-}^- \leq d_1^+ \leq \ldots \leq d_{n_+}^+$. Then, we can use the transformation $z = y\mathbb{O}$ to give

$$G_{\Gamma}(-V) = \int_{\{\Theta + z\mathbb{D}z^t/2 \leq -V\}} g(zz^t)dz,$$

and finally, we can use the transformation $w = |\mathbb{D}|^{1/2}z$ to give

$$G_{\Gamma}(-V) = \int_{\{|w_+|^2 - |w_-|^2 \leq -2(V+\Theta)\}} g(w|\mathbb{D}|^{-1}w^t)\frac{dw}{\sqrt{det(|\mathbb{D}|)}}, \tag{1}$$

where $w = (w_+, w_-)$ is the decomposition of $\mathbb{R}^n$ into the respective positive and negative subspaces of the eigenbasis for $\mathbb{C}^t\Gamma\mathbb{C}$. After changing the direction of the inequality, we obtain the following expression for $G(R)$, which will be the starting point for the discussion of our computational methods.

$$G(R) = \int_{\{|w_-|^2 - |w_+|^2 \geq R^2\}} g(w|\mathbb{D}|^{-1}w^t)\frac{dw}{\sqrt{det(|\mathbb{D}|)}}, \tag{2}$$

where $R^2 = 2(V + \Theta)$. The integration region is the *hyperboloid* defined by $|w_-|^2 - |w_+|^2 \geq R^2$. Our goal is to determine R a solution to $G(R) = \alpha$. Once we find R, we will have the approximate quadratic Value-at-Risk given by $V = R^2/2 - \Theta$.

4 Integration over Hyperboloids

4.1 The Normal Case

For many applications, the distribution g is a Normal distribution. In these cases,

$$G(R) = \int_{\{|w_-|^2 - |w_+|^2 \geq R^2\}} e^{-w|\mathbb{D}|^{-1}w^t/2} \frac{dw}{\sqrt{(2\pi)^n det(|\mathbb{D}|)}}.$$

Separating the w variables, we find

$$G(R) = \int_{\{|w_+|^2 \geq 0\}} e^{-w_+ D_+^{-1} w_+^t/2} \int_{\{|w_-|^2 \geq R^2 + |w_+|^2\}} e^{-w_- D_-^{-1} w_-^t/2} \frac{dw_-}{\sqrt{(2\pi)^{n_-} det(D_-)}} \frac{dw_+}{\sqrt{(2\pi)^{n_+} det(D_+)}}.$$

The inner integral for w_- can be efficiently computed using the algorithm described by Sheil and O'Muircheartaigh [11], so we define $H(R, r)$ by

$$H(R, r) = \int_{\{|w_-|^2 \geq R^2 + r^2\}} e^{-w_- D_-^{-1} w_-^t/2} \frac{dw_-}{\sqrt{(2\pi)_-^n det(D_-)}},$$

and let $w_+ = D_+^{\frac{1}{2}} z$ Then $G(R)$ can rewritten as

$$G(R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-xx^t/2} H(R, |zD_+z^t|) \frac{dz}{\sqrt{(2\pi)^{n_+}}}.$$

Integrals in this form can be approximated using methods described by Genz and Monahan [7].

4.2 The General Case

We need to determine approximations to integrals in the form

$$G(R) = \int_{\{|x|^2 - |y|^2 \geq R^2\}} \varphi(x, y) dy dz,$$

where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$. If we use the changes of variable: $y = r_2\xi_2$, $x = r_1\xi_1$, with $r_2 = |y|$ and $r_1 = |x|$, then $G(R)$ becomes

$$G(R) = \int_0^{\infty} r_2^{n_2-1} \int_{|\xi_2|=1} \int_{\sqrt{R^2+r_2^2}}^{\infty} r_1^{n_1-1} \int_{|\xi_1|=1} \varphi(r_1\xi_1, r_2\xi_2) d\sigma(\xi_1) dr_1 d\sigma(\xi_2) dr_2$$

We now have $G(R)$ defined in terms of a product of two integrals over hyper-spherical surfaces, defined by $|\xi_2| = 1$ and $|\xi_1| = 1$, and two radial integrals. The hyper-sphere surface integrals can be approximated using methods described in the paper by Genz [6]. If the surface and radial integrals are combined, then generalizations of the methods described by Genz and Monahan [7] can be used. Efficient approximation of the radial integrals will depend on information about the rate of decrease of the integrand φ for large values of r_1 and r_2.

5 Application Examples

We will distinguish 3 case in our analysis :

- n_-=0; if g is Normal,

$$G(R) = \int_{\{|w_+|^2 \leq R^2\}} e^{-w_+ D_+^{-1} w_+^t/2} \frac{dw_+}{\sqrt{(2\pi)^{n_+} det(D_+)}}.$$

  $G(R)$ can be efficiently computed using the Sheil and O'Muircheartaigh [11] algorithm, when $R^2 = -2(V + \Theta) \geq 0$.

- n_+=0; if g is Normal,

$$G(R) = \int_{\{|w_-|^2 \geq R^2\}} e^{-w_- D_-^{-1} w_-^t/2} \frac{dw_-}{\sqrt{(2\pi)^{n_-} det(D_-)}}.$$

  $G(R)$ can be efficiently computed using the Sheil and O'Muircheartaigh [11] algorithm.

- n_- and n_+ are both nonzero; g is Normal,

$$G(R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-xx^t/2} H(R, |zD_+z^t|) \frac{dz}{\sqrt{(2\pi)^{n_+}}}.$$

  Integrals in this form can be approximated using methods that are a combination of the Sheil and O'Muircheartaigh [11] and Genz and Monahan [7] algorithms.

5.1 An Example when n_+=0

We construct a Δ-hedged portfolio that contains n equities and n European call options on these equities from the *French CAC 40 Market*. The Price of the Portfolio is given by

$$\Pi(t, S(t)) = \sum_{i=1}^{n} [-C_i(t, S_i(t)) + \Delta^i \cdot S_i],$$

where S_i is an equity price i, with $S(t) = (S_1, \ldots, S_n)$, and $C_i(t, S_i(t))$ is the price of European call option i on equity i. Δ is known in the literature as a gradient portfolio sensitivity vector. Our portfolio has been chosen so that $\Delta = 0$, with $\Delta^i = \frac{\partial C_i}{\partial S_i}(S_i(0))$, and $\Delta = (\Delta^1, \ldots, \Delta^n)$. The exercise price of each European call option is given in the following table for an example where $n = 9$.:

Table 1: Data for Nine CAC 40 European Call Options

	Exercise Price	Interest Rate	Maturity	Underlying Price
BNPPARIBAS	44.26	0.1	3 month	39.75
BOUYGUES	23.49	0.1	3 month	27.30
CAP GEMINI	34.71	0.1	3 month	24.00
CREDIT AGRICOLE	17.36	0.1	3 month	14.80
DEXIA	11.5	0.1	3 month	9.38
LOREALL	61.85	0.1	3 month	62.90
TF1	26.38	0.1	3 month	22.02
THOMSON	15.22	0.1	3 month	17.13
VIVENDI	16.19	0.1	3 month	17.00

Using the above data with the exponential moving weighted average (EMWA), we obtained the following Σ:

$$\Sigma = \begin{pmatrix} 0.0017 & -0.0001 & 0.0012 & 0.0005 & 0.0008 & 0.0008 & 0.0008 & 0.0002 & 0.0002 \\ -0.0001 & 0.0009 & 0.0005 & -0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0006 & 0.0005 \\ 0.0012 & 0.0005 & 0.0038 & 0.0006 & 0.0011 & 0.0008 & 0.0014 & 0.0006 & 0.0007 \\ 0.0005 & -0.0001 & 0.0006 & 0.0006 & 0.0002 & 0.0004 & 0.0004 & 0.0001 & 0.0001 \\ 0.0008 & 0.0000 & 0.0011 & 0.0002 & 0.0015 & 0.0007 & 0.0008 & 0.0000 & 0.0002 \\ 0.0008 & 0.0000 & 0.0008 & 0.0004 & 0.0007 & 0.0011 & 0.0006 & 0.0004 & 0.0005 \\ 0.0008 & 0.0000 & 0.0014 & 0.0004 & 0.0008 & 0.0006 & 0.0013 & 0.0000 & 0.0001 \\ 0.0002 & 0.0006 & 0.0006 & 0.0001 & 0.0000 & 0.0004 & 0.0000 & 0.0019 & 0.0007 \\ 0.0002 & 0.0005 & 0.0007 & 0.0001 & 0.0002 & 0.0005 & 0.0001 & 0.0007 & 0.0029 \end{pmatrix}$$

The matrix Γ is a diagonal matrix with diagonal entries given by

$$d = (-116.4889, -33.4063, -11.8389, -21.1723, -11.9582, -161.2178, -34.7884, -19.7664, -27.2993)$$

and $\Theta = -31.2689$. The eigenvalues of the $\mathbb{D}$ matrix are given by the vector

$$e = (-.05025, -.1456, -.0605, -.0424, -.0231, -.0.0053, -.0101, -.0154, -.0131).$$

The following Table provides some R and V values that were found as numerical solutions to the equation $G(R) = \alpha$, for selected α's.

α	0.05	0.025	0.01
R	0.9160	1.0038	1.1128
V	31.6883	31.7727	31.8881

5.2 Example with $n_- > 0$ and $n_+ > 0$

We consider a portfolio that contains call options and put options on equities, so that price of the portfolio at time t, is given by:

$$\Pi(t) = \sum_{i=1}^{5}[C_i(t, S_i(t)) - \delta_i S_i(t)] - \sum_{j=6}^{10} P_j(t, S_j(t)) - (\delta_j - 1)S_j(t)]$$

The prices of each of the options are taken from data from the *French CAC 40 Market*, and are given in the following table:

Table 2: Data for Ten CAC 40 European Call and Put Options

	Exercize Price	Interest Rate	Maturity	Underlying Price
Call-BNPPARIBAS	30.00	0.05	3 months	39.75
Call-BOUYGUES	19.00	0.05	3 months	27.30
Call-CAP GEMINI	20.00	0.05	3 months	24.00
Call-CREDIT AGRICOLE	10.50	0.05	3 months	14.80
Call-DEXIA	9.00	0.05	3 months	9.38
Call-LOREALL	40.00	0.05	3 months	62.90
Put-SOCIETEGENERALE	50	0.05	3 months	64.00
Put-TF1	18.00	0.05	3 months	22.02
Put-THOMSON	9.00	0.05	3 months	17.13
Put-VIVENDI	9.00	0.05	3 months	17.00

In this example, using the three month historical data for the ten CAC 40 equities, with the exponential moving weighted average (EMWA) and $\lambda = 0.94$, we obtained the following Σ.

$$\Sigma = \begin{pmatrix} 0.0016 & -0.0001 & 0.0012 & 0.0005 & 0.0008 & 0.0007 & -0.0000 & 0.0008 & 0.0002 & 0.0002 \\ -0.0001 & 0.0008 & 0.0004 & -0.0001 & 0.0000 & 0.0000 & 0.0004 & -0.0000 & 0.0005 & 0.0005 \\ 0.0011 & 0.0004 & 0.0035 & 0.0006 & 0.0010 & 0.0007 & 0.0004 & 0.0013 & 0.0006 & 0.0006 \\ 0.0005 & -0.0001 & 0.0006 & 0.0005 & 0.0002 & 0.0004 & -0.0001 & 0.0003 & 0.0001 & 0.0001 \\ 0.0008 & 0.0000 & 0.0010 & 0.0002 & 0.0015 & 0.0007 & -0.0000 & 0.0008 & 0.0000 & 0.0002 \\ 0.0007 & 0.0000 & 0.0007 & 0.0004 & 0.0007 & 0.0010 & -0.0002 & 0.0005 & 0.0003 & 0.0004 \\ -0.0000 & 0.0004 & 0.0004 & -0.0001 & -0.0000 & -0.0002 & 0.0015 & -0.0002 & 0.0008 & -0.0001 \\ 0.0008 & -0.0000 & 0.0013 & 0.0003 & 0.0008 & 0.0005 & -0.0002 & 0.0012 & 0.0000 & 0.0001 \\ 0.0002 & 0.0005 & 0.0006 & 0.0001 & 0.0000 & 0.0003 & 0.0008 & 0.0000 & 0.0018 & 0.0007 \\ 0.0002 & 0.0005 & 0.0006 & 0.0001 & 0.0002 & 0.0004 & -0.0001 & 0.0001 & 0.0007 & 0.0026 \end{pmatrix}.$$

The matrix Γ is a diagonal matrix with diagonal

$$d = (24.186, 8.6269, 21.7320, 4.3111, 15.4949, -4.5815, -82.2915, -22.2079, -1.2957, -1.2822),$$

and $\Theta = -3.8596$. The eigenvalues of the $\mathbb{D}$ matrix are given by the vector

$$e = (-0.1251, -0.0115, -0.0030, -0.0014, -0.0006, 0.0014, 0.0092, 0.0124, 0.0290, 0.1271).$$

The following Table provides some R and V values that were found as numerical solutions to the equation $G(R) = \alpha$, for selected α's.

α	0.05	0.025	0.01
R	0.6069	0.7176	0.8455
V	4.0438	4.1171	4.2166

6 Conclusions

We have considered an application in the domain of Risk Management to estimate the Value-at-Risk of the portfolio of an option. We have reduced the problem to a multiple integral over a hyperboloid. This type of integral can be approximated using techniques described by Genz and Monahan [7] and Sheil and O'Muircheartaigh [11].

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